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A convergent scheme for one-loop evolutions of the Yukawa couplings in the MSSM

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Abstract. Integrated forms of the one-loop evolution equations are given for the Yukawa couplings in the MSSM, valid for any value of $\tan\beta$, generalizable to virtually any number of Yukawa fermions, and including all gauge couplings. These forms turn out to have nice mathematical convergence properties which we prove, and we determine the ensuing convergence criteria. Furthermore, they allow one to write down general sufficient and necessary conditions to avoid singularities in the evolution of the Yukawa couplings over physically relevant energy ranges. We also comment briefly on the possible use of these features for physics issues and give a short numerical illustration.

1 Introduction

Large Yukawa couplings often play an important role in spontaneous electroweak symmetry breaking models such as the standard model, its extensions, and also in some alternatives to it. The existence of infrared (IR) attractive fixed points [1] and effective IR fixed points [2] in such a regime can be a benchmark in selecting the phenomenologically viable theories in view of the measured mass of the top quark. Such considerations clearly favoured the minimal supersymmetric extension of the standard model (MSSM) [3], with its upper bound on the top mass, $m_t \leq (190\text{–}200\,\mathrm{GeV})\sin\beta$ [4] over alternatives such as [5].

In supersymmetry there is of course much more to say. In the minimal supersymmetric extension of the standard model (MSSM) [6], the spontaneous breaking of the electroweak symmetry is generically driven by the large top quark mass in association with a soft supersymmetry breaking sector [3]. Such a possibility can be looked at as the phenomenological aspect of a [still to be discovered] deep connection between the origin of supersymmetry breaking and that of the electroweak symmetry breaking. Meanwhile, it helps in correlating theoretically the very many free parameters of the MSSM, leading to quantitative estimates of the supersymmetric partners spectrum, which are of prime importance in guiding the experimental search for supersymmetry, and relating (at least qualitatively) the physics over as many orders of magnitudes as there are between the GUT scales and the electroweak scale.

Of course, the key point in all the above issues is the way the various parameters "run", as dictated by the renormalization group equations. The RGEs are, in general, complicated coupled differential equations already at the one-loop level and one usually resorts to numerical methods to solve them [7]. However, analytical solutions would be desirable for several reasons (besides the obvious one of allowing a better control over the structure of the running). The radiative breaking of $SU_{\mathrm{c}}(3) \times SU_{\mathrm{L}}(2) \times U_Y(1)$ and the structure of the vacuum are controlled by the effective potential (EP) of the theory. The theoretical improvement of the functional form of the EP needs the analytical form of the running of involved quantities such as masses, couplings and fields which enter the game. If the analytical form of the EP beyond the tree level were known exactly, or at least in an RG improved form beyond the naive loop corrections, one could hope to determine the relations the initial values of the parameters should fulfill, to break the electroweak symmetry at the right energy scale and avoid in the same time color and charge breaking vacua[1]. Another important application of the analytical solutions is the determination of the effective behaviour in the low energy regime (infrared regions) independently (but within a domain) of the actual values in the ultraviolet, such as the top Yukawa coupling effective fixed point [2], or the triviality bounds on the Higgs mass.

Analytical solutions of the complete set of RGEs in the MSSM were known to one-loop order for small $\tan\beta$, strictly speaking in the case all Yukawa couplings are put to zero except for the top quark. Actually the structure of the coupled equations is such that a necessary condition to solve them entirely is to be able to solve first for the Yukawa couplings. This is of course not enough, and initially one assumed also universality of the soft SUSY breaking terms at the GUT scale in order to solve for those quantities too. This assumption can however be re-

[1] In the absence of such a knowledge, due partly (in the RG improvement program) to the existence of many different mass scales, one relies on rough approximations hoping that they encompass the leading behaviour [8]

laxed [10] but still for small $\tan\beta$. It is thus natural to try to find exact solutions for the Yukawa sector for any value of $\tan\beta$, for comparable top and bottom Yukawa couplings, and also bringing in the game the τ lepton Yukawa coupling as well in order to cope with the case of b–τ unification [11]. Some attempts have been made (for instance in [12]) to generally solve the top–bottom Yukawa system which leads to implicit solutions provided one neglects the $U(1)_Y$ gauge coupling (see also [13]).

In the present paper we study some properties of the runnings of the Yukawa couplings as dictated by the RGEs to one loop. The first aim is to provide suitable expressions for the exact solutions, which we call "integrated forms". Although these expressions do not appear in closed forms, they are especially convenient because, if one insists on making them explicit, they come out as continued integrated fractions the convergence of which can be kept under control. Although our main results are a priori valid for any gauge theory with an arbitrarily extended Yukawa sector (including the special case of the standard model), we restrict most of the discussions and further illustrations to the case of the MSSM. We will give integrated forms, valid for any number of Yukawa couplings, of the general explicit solutions corresponding to the coupled renormalization group equations which read in the MSSM [14]

Gauge couplings (g_i with $i = 1, 2, 3$ and n_g the generation number)

$$\frac{\mathrm{d}g_i}{\mathrm{d}t} = \frac{1}{32\pi^2} b_i g_i^3 \quad \text{with} \quad b_1 = -1 - \frac{10}{3} n_{\mathrm{g}},$$
$$b_2 = 5 - 2n_{\mathrm{g}}, \quad b_3 = 9\text{–}2n_{\mathrm{g}}. \tag{1.1}$$

Yukawa couplings ($i = 1, 2, 3$ generations)

$$\frac{\mathrm{d}Y_u^i}{\mathrm{d}t} = -\frac{Y_u^i}{32\pi^2}\Bigg[3(Y_u^i)^2 + 3\sum_{k=\mathrm{gen}} (Y_u^k)^2 + (Y_d^i)^2 - \left(\frac{13}{9}g_1^2 + 3g_2^2 + \frac{16}{3}g_3^2\right)\Bigg], \tag{1.2}$$

$$\frac{\mathrm{d}Y_d^i}{\mathrm{d}t} = -\frac{Y_d^i}{32\pi^2}\Bigg[3(Y_d^i)^2 + (Y_u^i)^2 + \sum_{k=\mathrm{gen}} \{3(Y_d^k)^2 + (Y_l^k)^2\} - \left(\frac{7}{9}g_1^2 + 3g_2^2 + \frac{16}{3}g_3^2\right)\Bigg], \tag{1.3}$$

$$\frac{\mathrm{d}Y_l^i}{\mathrm{d}t} = -\frac{Y_l^i}{32n1.1\pi^2}\Bigg[3(Y_l^i)^2 + \sum_{k=\mathrm{gen}} \{(Y_l^k)^2 + 3(Y_d^k)^2\} - 3(g_1^2 + g_2^2)\Bigg]. \tag{1.4}$$

Here the evolution parameter t is defined by $t = \mathrm{Log}(M_U^2/Q^2)$ where M_U denotes some initial scale. Note that since a gauge coupling unification condition is not essential in the present study, we write the RGE equations in terms of the low energy $SU(3)_{\mathrm{c}} \times SU(2)_{\mathrm{L}} \times U(1)_Y$ gauge couplings, respectively g_3, g_2 and g_1. Note also that we assume here, and throughout the paper, flavour conserving (diagonal) Yukawa matrices.

The rest of the paper is organized as follows. In Sect. 2 we recall the known solution for large top Yukawa coupling and give the integrated form of the general solution valid for any value of the top and bottom Yukawas. We then generalize those integrated forms to any number of Yukawa couplings, in particular to the top–bottom–τ system. In Sect. 3 we give a proof of the convergence of these forms in both the top–bottom and the top–bottom–τ case. Section 4 is devoted to the question of avoiding Landau poles in the Yukawa runnings. There we give a generalization to the top–bottom case of some well-known bounds, and establish necessary and sufficient conditions. Preliminary applications and comments are made in Sect. 5 and conclusions and an outlook are given in Sect. 6. An Appendix contains some detailed proofs and technical material.

2 Integrated form of the Yukawa coupling RGEs

2.1 Large top quark Yukawa solutions: a reminder

We are interested here in (1.2)–(1.4). They can be treated independently of the rest of the system, especially from the gauge couplings for which the running is determined a priori via (1.1). (This is no more true at two-loop order where the gauge and Yukawa equations become highly interwound.) When all Yukawa couplings except Y_t are neglected, (1.3) and (1.4) become trivial while (1.2) becomes of the Bernoulli type in the variable $y_t \equiv Y_t^2$:

$$\frac{\mathrm{d}}{\mathrm{d}t} y_t = f_1(t) y_t + b y_t^2, \tag{2.1}$$

where

$$f_1(t) = \frac{1}{16\pi^2}\left(\frac{16}{3}g_3^2 + 3g_2^2 + \frac{13}{9}g_1^2\right), \quad b = -\frac{6}{16\pi^2}, \tag{2.2}$$

and it is easily solved to give [15, 9]

$$y_t(t) = \frac{y^0 E(t)}{1 - b y^0 \int_0^t E(t')\mathrm{d}t'}, \tag{2.3}$$

where

$$E(t) = \mathrm{e}^{\int_0^t f_1(t')\mathrm{d}t'} \quad \text{and} \quad y^0 = Y_t^2(t=0). \tag{2.4}$$

2.2 top–bottom case

In the more general case where both Y_t and Y_b are kept in the game, but neglecting all other Yukawa couplings, (1.2) and (1.3) read, after the change of variables $y_t \equiv Y_t^2, y_b \equiv Y_b^2$,

$$\frac{\mathrm{d}}{\mathrm{d}t} y_t = f_1(t) y_t + a y_b y_t + b y_t^2,$$
$$\frac{\mathrm{d}}{\mathrm{d}t} y_b = f_2(t) y_b + a y_b y_t + b y_b^2, \tag{2.5}$$

where $f_1(t)$ and b are given in (2.2) and

$$f_2(t) = \frac{1}{16\pi^2}(\frac{16}{3}g_3^2 + 3g_2^2 + \frac{7}{9}g_1^2), \qquad a = -\frac{1}{16\pi^2}. \quad (2.6)$$

As far as we know, the system (2.5) is not treated in standard text books, and although it looks simple at first sight, we could not find a systematic way of relating it to a standard form[2]. It is also relatively easy to solve the system up to first order in Y_b in the region $Y_t \gg Y_b$. This is already an improvement of the known solutions with $Y_b \sim 0$. It extends the numerical validity much further than $\tan\beta \simeq 10$.

More importantly, this approximate solution gives a valuable hint for the structure of a suitable integrated form which can then be found by sheer guess and will be written down below. But first, the approximate solution can be obtained in the form $y_t(t) = \tilde{y}_t(t)+\delta(t)$ where $\tilde{y}_t(t)$ is given by (2.3) and when necessary we linearize the equations in the regime $y_b(t), |\delta(t)| \ll 1$. One then finds for y_b

$$y_b(t) = \frac{y_b^0 E_{21}(t)}{1 - b y_b^0 \int_0^t E_{21}(t')\mathrm{d}t'}, \quad (2.7)$$

where

$$E_{21}(t) = \frac{E_2(t)}{(1 - b y_t^0 \int_0^t E_1(t')\mathrm{d}t')^{a/b}}, \quad (2.8)$$

$$E_i(t) = \mathrm{e}^{\int_0^t f_i(t')\mathrm{d}t'} \quad i = 1, 2, \quad (2.9)$$

and a slightly more complicated expression for y_t, given in Appendix B. A little thinking then leads to the following form of the exact "solution" we were looking for:

$$y_t(t) = \frac{y_t^0 E_{12}(t)}{1 - b y_t^0 \int_0^t E_{12}(t')\mathrm{d}t'}, \quad (2.10)$$

$$y_b(t) = \frac{y_b^0 E_{21}(t)}{1 - b y_b^0 \int_0^t E_{21}(t')\mathrm{d}t'}, \quad (2.11)$$

where

$$E_{12}(t) = \frac{E_1(t)}{(1 - b y_b^0 \int_0^t E_{21}(t')\mathrm{d}t')^{a/b}}, \quad (2.12)$$

$$E_{21}(t) = \frac{E_2(t)}{(1 - b y_t^0 \int_0^t E_{12}(t')\mathrm{d}t')^{a/b}}, \quad (2.13)$$

and $y_t^0 \equiv Y_t^2(t=0), y_b^0 \equiv Y_b^2(t=0)$ are any initial conditions. The reader can easily check that the solutions (2.10) and (2.11) *exactly* satisfy (2.5) without any restriction or assumption about the magnitudes of the Yukawa couplings, i.e. for any value of $\tan\beta$. They formally resemble (2.3) of which they are a generalization. Of course, although our solutions, y_t, y_b are now explicit in terms of E_{12} and E_{21}, the latter are given only implicitly by (2.12) and (2.13), which appear as coupled nonlinear integral equations. Therefore, the procedure is useful only if it provides us with a systematic (and hopefully quick) way to solve these equations within a given accuracy. It will be shown in Sect. 3 that mere iterations achieve this goal. In fact, such iterations correspond to the truncations of the "continued integrated fractions" which naturally emerge as *formal* solutions of (2.12) and (2.13), e.g.: (see (2.14) on top of the page)

[2] The situation would be much simpler if $f_1(t) = f_2(t)$, in which case the equations can be solved by quadrature after some change of variables, leading though only to implicit solutions involving some hypergeometric functions [12]

2.3 Arbitrary number of Yukawa fermions

In fact, the above solutions are easily generalized to include any number of leptons and quarks. For instance, if one includes in the game a third Yukawa coupling, then (1.2), (1.3 and (1.4) take the following form:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}y_1 &= f_1(t)y_1 + a_{11}y_1^2 + a_{12}y_1y_2 + a_{13}y_1y_3,\\
\frac{\mathrm{d}}{\mathrm{d}t}y_2 &= f_2(t)y_2 + a_{22}y_2^2 + a_{21}y_2y_1 + a_{23}y_2y_3,\\
\frac{\mathrm{d}}{\mathrm{d}t}y_3 &= f_3(t)y_3 + a_{33}y_3^2 + a_{31}y_3y_1 + a_{32}y_3y_2.
\end{aligned}$$

The exact solution reads:

$$\begin{aligned}
y_1 &= \frac{y_1^0 u_1}{1 - a_{11}y_1^0 \int u_1},\\
y_2 &= \frac{y_2^0 u_2}{1 - a_{22}y_2^0 \int u_2},\\
y_3 &= \frac{y_3^0 u_3}{1 - a_{33}y_3^0 \int u_3},
\end{aligned}$$

where u_1, u_2 and u_3 are defined through the implicit system

$$\begin{aligned}
u_1 &= \frac{E_1}{(1 - a_{22}y_2^0 \int u_2)^{a_{12}/a_{22}}(1 - a_{33}y_3^0 \int u_3)^{a_{13}/a_{33}}},\\
u_2 &= \frac{E_2}{(1 - a_{11}y_1^0 \int u_1)^{a_{21}/a_{11}}(1 - a_{33}y_3^0 \int u_3)^{a_{23}/a_{33}}},\\
u_3 &= \frac{E_3}{(1 - a_{11}y_1^0 \int u_1)^{a_{31}/a_{11}}(1 - a_{22}y_2^0 \int u_2)^{a_{32}/a_{22}}},
\end{aligned} \quad (2.15)$$

and $\int u_j$ stands for $\int_0^t \mathrm{d}t' u_j(t')$.

In the interesting case of the top–bottom–τ system with $y_t \equiv y_1$, $y_b \equiv y_2$ and $y_\tau \equiv y_3$ one has in the MSSM

$$\begin{aligned}
&a_{11} = a_{22} = -\frac{6}{16\pi^2}; \quad a_{33} = -\frac{4}{16\pi^2},\\
&\frac{a_{12}}{a_{22}} = \frac{a_{21}}{a_{11}} = \frac{1}{6}; \quad \frac{a_{31}}{a_{11}} = \frac{a_{13}}{a_{33}} = 0,\\
&\frac{a_{23}}{a_{33}} = \frac{1}{4}; \quad \frac{a_{32}}{a_{22}} = \frac{1}{2},\\
&f_3(t) = \frac{3}{16\pi^2}(g_1^2 + g_2^2); \quad E_3(t) = \mathrm{e}^{\int_0^t f_3(t')\mathrm{d}t'},
\end{aligned} \quad (2.16)$$

$$E_{12}(t) = \cfrac{E_1(t)}{(1 - by_b^0 \int_0^t \cfrac{E_2(t_1)\mathrm{d}t_1}{(1 - by_t^0 \int_0^{t_1} \cfrac{E_1(t_2)\mathrm{d}t_2}{(1 - by_b^0 \int_0^{t_2} \cfrac{E_2(t_3)\mathrm{d}t_3}{(1 - by_t^0 \int_0^{t_3} \cfrac{E_1(t_4)\mathrm{d}t_4}{\ddots})^{a/b}})^{a/b}})^{a/b}})^{a/b}}. \tag{2.14}$$

and $f_{1,2}(t), E_{1,2}(t)$ as previously. It is interesting to note that in this case u_τ and u_t are directly related via

$$\frac{u_\tau}{E_3} = \left(\frac{u_t}{E_1}\right)^3, \tag{2.17}$$

a reflection of the fact that, in the MSSM, the running of y_t and y_τ at one-loop order are mutually affected only indirectly through the running of y_b ($a_{31} = a_{13} = 0$), at variance with the case of the non-supersymmetric standard model (SM).

Finally, the extension to more than three Yukawa couplings will not play any role in the present paper. Nevertheless, we give it for the sake of completeness in Appendix C.

3 Proof of convergence

3.1 The top–bottom case

In this section we make a mathematical digression to study some useful properties of our solutions. Even though (2.12) and (2.13) give E_{ij} only implicitly, they enjoy the property of defining a *contraction mapping*. This is about all that one needs to give a rigorous proof for the existence and uniqueness of the E_{ij}, and thus of the existence and uniqueness of the solutions given in (2.10) and (2.11). This proof will also be of practical use. It provides us with a criterion for the convergence of the truncated forms of (2.14) towards the exact solution, and the rate of this convergence can be controlled so that a very good approximation will be obtained with a few iterations (or even just one).

For the sake of completeness, we recall here in simple terms the conditions required for a contraction mapping, and then prove that they are indeed satisfied in our case. Let us define

$$U_1(t) = \frac{E_{12}(t)}{E_1(t)}, \qquad U_2(t) = \frac{E_{21}(t)}{E_2(t)}, \tag{3.1}$$

and think of U_1 and U_2 as forming a vector

$$\vec{U}(t) = \begin{pmatrix} U_1(t) \\ U_2(t) \end{pmatrix}, \tag{3.2}$$

in some space $\mathcal{E}_T$ where the evolution parameter t remains in the interval $0 \le t \le T$ for a given value of T. Then (2.12) and (2.13) restrict the $U_i's$ to the positive region $U_i(t) \ge 0$ (provided one stays far from the Landau poles), and, furthermore, define a mapping in this region, $\mathcal{A} : \vec{U} \mapsto \vec{U}'$ through

$$U_1'(t) = \frac{1}{(1 - by_2^0 \int_0^t E_2(t')U_2(t')\mathrm{d}t')^{a/b}}, \qquad U_2'(t) = \frac{1}{(1 - by_1^0 \int_0^t E_1(t')U_1(t')\mathrm{d}t')^{a/b}}. \tag{3.3}$$

The idea now is to show that the mapping $\mathcal{A}$ shrinks uniformly, at each iteration, the "distance" between any two vectors in $\mathcal{E}_T$ (subject to the condition $U_i(t) \ge 0$). More precisely we will prove that there exists a positive constant number $K_T < 1$ such that the following inequality is satisfied

$$\| \vec{U'} - \vec{V'} \| \le K_T \| \vec{U} - \vec{V} \|, \tag{3.4}$$

for any pair of vectors $(\vec{U}, \vec{V})$ belonging to $\mathcal{E}_T$ and satisfying $U_i, V_i \ge 0$ ($i = 1, 2$). Here $\| \,.\, \|$ is defined by

$$\| \vec{U} \| = \max\{ \sup_{0 \le t \le T} | U_1(t) |, \sup_{0 \le t \le T} | U_2(t) | \}. \tag{3.5}$$

Then, according to the "contraction mapping principle", the existence of the (unique) solution of (2.12) and (2.13) in $\mathcal{E}_T$ is guaranteed, and the nth iteration $\mathcal{A}^n \vec{U}$ approaches this solution at least as fast as K_T^n.

To prove (3.4) one writes the following sequence of inequalities

$$\begin{aligned} &| U_i'(t) - V_i'(t) | \\ &\le \left\{ y_j^0 \int_0^t \mathrm{d}t' E_j(t') \mid U_j(t') - V_j(t') \mid \right\} \\ &\quad \Bigg/ \left\{ \left[16\pi^2 + 6y_j^0 \min\left\{ \int_0^t \mathrm{d}t' E_j(t')U_j(t'), \right.\right.\right. \\ &\quad \left.\left.\left. \int_0^t \mathrm{d}t' E_j(t')V_j(t') \right\} \right]^{7/6} \right\} \\ &\le \frac{y_j^0}{16\pi^2} \sup_{0 \le \tau \le T} | U_j(\tau) - V_j(\tau) | \int_0^T \mathrm{d}t' E_j(t') \\ &\le \frac{y_j^0}{16\pi^2} \| \vec{U} - \vec{V} \| \int_0^T \mathrm{d}t' E_j(t'), \end{aligned} \tag{3.6}$$

valid for $i \neq j$ with $i, j = 1, 2$ with $y_1 \equiv y_t, y_2 \equiv y_b$ and where we plugged the actual values of the coefficients a, b in (2.5). The first inequality in (3.6) is derived from (3.3) and from the inequality,

$$\mid \left(\frac{1}{1+\alpha}\right)^c - \left(\frac{1}{1+\beta}\right)^c \mid \leq c \frac{\mid \alpha - \beta \mid}{(1 + \min\{\alpha, \beta\})^{c+1}}, \quad (3.7)$$

valid for any α, β, larger than -1 and $c > 0$ (here $c = 1/6$), the second from the positivity of U_i, V_i, E_i and y_i^0, and the third from the definition (3.5). Equation (3.6) immediately leads to (3.4) with

$$K_T = \frac{1}{16\pi^2} \max \left\{ y_t^0 \int_0^T \mathrm{d}t E_1(t), y_b^0 \int_0^T \mathrm{d}t E_2(t) \right\}. \quad (3.8)$$

The convergence condition $K_T < 1$ is easily met even when T is large enough to encompass the whole evolution range from the GUT scale to the M_Z scale. For instance, if $T \approx 66$ and $\alpha_{\mathrm{GUT}}^{-1} \approx 25$, one needs $Y_t^0, Y_b^0 \lesssim O(\pi)$ where the Y_i^0 are the GUT scale values of the Yukawa couplings, to ensure convergence. These conditions are naturally met within the perturbative regime. We should stress, however, that these are only sufficient conditions.

3.2 The top–bottom–τ case

As we said previously the exact solutions (2.15) and (2.15) to the generalized equations with three Yukawa couplings or more, is a direct generalization of (2.10) and (2.11). Similarly, the proof of convergence goes essentially along the same lines as in the previous section, when supplemented with the inequality.

$$\left| \frac{1}{(1+\alpha_1)^a} \frac{1}{(1+\beta_1)^b} - \frac{1}{(1+\alpha_2)^a} \frac{1}{(1+\beta_2)^b} \right| \leq \frac{a \mid \alpha_1 - \alpha_2 \mid}{(1 + \min\{\alpha_1, \alpha_2\})^{a+1}} + \frac{b \mid \beta_1 - \beta_2 \mid}{(1 + \min\{\beta_1, \beta_2\})^{b+1}},$$

valid for any $a, b, \alpha_i, \beta_i > 0$. Defining a mapping through (2.15), one finds the convergence criterion

$$K_T = \max_{(ijk)\text{circ. perm. of } (123)} \times \left\{ -a_{ij} y_j^0 \int_0^T E_j - a_{ik} y_k^0 \int_0^T E_k \right\} < 1. \quad (3.9)$$

In the top–bottom–τ case it reads

$$\begin{aligned} K_T &= \frac{1}{16\pi^2} \max \left\{ y_b^0 \int_0^T E_2; \;\; y_t^0 \int_0^T E_1 + y_\tau^0 \int_0^T E_3; \right. \\ &\quad \left. 3 y_b^0 \int_0^T E_2 \right\} \\ &= \frac{1}{16\pi^2} \max \left\{ y_t^0 \int_0^T E_1 + y_\tau^0 \int_0^T E_3; \right. \\ &\quad \left. 3 y_b^0 \int_0^T E_2 \right\} < 1. \end{aligned} \quad (3.10)$$

We see that the sufficient convergence criterion can become more severe in this case by about a factor 3 in the regime $y_t^0 \sim y_b^0 \sim y_\tau^0$.

4 Avoiding Landau poles

One question that can be clearly answered with the knowledge of analytical solutions is how to determine the conditions which guarantee that the values of the Yukawa couplings at some low energy scale, say the electroweak scale, remain consistent with a Landau "pole" free theory up to a given high energy scale, typically a grand unification scale. The answer would be trivial if one starts from the high energy scale Yukawa coupling and runs down. Indeed in this case, it is clear from the general form of the solutions, (2.10) and (2.11) and the fact that $b < 0, y^0 (\equiv {Y^0}^2) \geq 0$ and $E(t)_{ij} \geq 0$, that one does not hit a Landau pole all the way down below the initial scale[3]. The situation is more complicated if one starts from some Yukawa coupling values at a low scale and tries to run upwards to determine the corresponding values at a GUT scale. This is a phenomenologically typical situation if a model-independent reconstruction of the fundamental parameters is to be carried out, starting from the experimental input.

Moreover, since in the vicinity of such Landau poles the Yukawa couplings become very large, the conditions for avoiding these poles correspond in some cases to effectively attractive fixed points, such as the celebrated relation

$$m_{\mathrm{top}} \approx (190\text{–}200\,\mathrm{GeV}) \sin\beta, \quad (4.1)$$

in the MSSM valid when all Yukawa couplings are neglected in comparison to the that of the top quark [4]. When the Yukawa couplings are of comparable values such a correspondence becomes more involved, as can be seen for instance from the dependence on y_t^0 and y_b^0 in (2.14) [See also the discussion in Sect. 5.2].

To illustrate the case, we start first with the solution for small $\tan\beta$, i.e. $Y_t^0 \gg Y_b^0 \sim 0$, given in (2.3). Writing it in the form

$$y^0 = \frac{y_t(t)}{E(t) + b y_t(t) \int_0^t E(t')\mathrm{d}t'}, \quad (4.2)$$

one sees immediately that $y_t(t)$ should satisfy

$$y_t(t) < -\frac{E(t)}{b \int_0^t E(t')\mathrm{d}t'}, \quad (4.3)$$

for any value of $t > 0$ in order to be consistent with the positivity of y^0 and $y_t(t)$. Ensuring this positivity automatically avoids the Landau pole (we rely here on the fact that $b < 0$, see (2.2)). Thus, contrary to the initial

[3] Note that here we are only interested in poles which can occur explicitly in the Yukawa couplings. In practice one any way stays far from gauge coupling Landau poles, given the running range relevant to our discussion

value at a high scale $t = 0$, values of y_t cannot be arbitrarily chosen at lower scales, the maximum allowed in this case being simply given by (4.3). With this in mind, it is straightforward to completely eliminate the dependence on y^0 in the running to get

$$y_t(t) = \frac{y_t(t_0)E(t;t_0)}{1 - by_t(t_0)\int_{t_0}^{t} E(t';t_0)\mathrm{d}t'}, \tag{4.4}$$

where

$$E(t;t_0) \equiv \frac{E(t)}{E(t_0)}. \tag{4.5}$$

Here $y_t(t_0)$ is any initial value satisfying (4.3) for $t = t_0$.

In the more general case, when y_b is not neglected, one can also write

$$\begin{aligned} y_b(t) &= \frac{y_b(t_0)E_{21}(t;t_0)}{1 - by_b(t_0)\int_{t_0}^{t} E_{21}(t';t_0)\mathrm{d}t'}, \\ y_t(t) &= \frac{y_t(t_0)E_{12}(t;t_0)}{1 - by_t(t_0)\int_{t_0}^{t} E_{12}(t';t_0)\mathrm{d}t'}, \end{aligned} \tag{4.6}$$

with

$$\begin{aligned} E_{12}(t;t_0) &= \frac{E_1(t;t_0)}{\left(1 - by_b(t_0)\int_{t_0}^{t} E_{21}(t';t_0)\mathrm{d}t'\right)^{a/b}}, \\ E_{21}(t;t_0) &= \frac{E_2(t;t_0)}{\left(1 - by_t(t_0)\int_{t_0}^{t} E_{12}(t';t_0)\mathrm{d}t'\right)^{a/b}}, \end{aligned} \tag{4.7}$$

where

$$E_j(t;t_0) = \mathrm{e}^{-\int_t^{t_0}\mathrm{d}t' f_j(t')} = \frac{E_j(t)}{E_j(t_0)}. \tag{4.8}$$

Note that here $t_0 \geq t$ corresponds to an initial energy scale *lower* than the running scale corresponding to t. In this case, however, it is no more possible to easily determine a sufficient and necessary condition (if any) on $y_b(t_0)$ and $y_t(t_0)$ to avoid the Landau pole. Indeed, the sufficient and necessary conditions for a non-singular running in the interval $[T, t_0]$ read

$$1 - by_b(t_0)\int_{t_0}^{T} E_{21}(t';t_0)\mathrm{d}t' > 0, \tag{4.9}$$

$$1 - by_t(t_0)\int_{t_0}^{T} E_{12}(t';t_0)\mathrm{d}t' > 0, \tag{4.10}$$

as easily seen from (4.6). However, E_{12} and E_{21} themselves depend on y_t^0 and y_b^0, so that the above conditions are highly implicit in y_t^0 and y_b^0.

Instead, one can immediately determine some *necessary* conditions, and, with some extra work, also some *sufficient* ones. We list these conditions below and refer the reader to the Appendix for the detailed proofs.

The necessary conditions: one can write a tower of pair of inequalities, each pair being a necessary condition weaker than the subsequent one,

$$\begin{cases} y_t(t_0) < \dfrac{E_1(t_0)}{|b|\int_T^{t_0}\mathrm{d}tE_1(t)}, \\[2ex] y_b(t_0) < \dfrac{E_2(t_0)}{|b|\int_T^{t_0}\mathrm{d}tE_2(t)}, \end{cases} \tag{4.11}$$

$$\begin{cases} y_t(t_0) < \dfrac{E_1(t_0)}{|b|\int_T^{t_0}\dfrac{\mathrm{d}t_1E_1(t_1)}{\left(1-|b|y_b(t_0)\int_{t_1}^{t_0}E_2(t_2;t_0)\mathrm{d}t_2\right)^{a/b}}}, \\[2ex] y_b(t_0) < \dfrac{E_2(t_0)}{|b|\int_T^{t_0}\dfrac{\mathrm{d}t_1E_2(t_1)}{\left(1-|b|y_t(t_0)\int_{t_1}^{t_0}E_1(t_2;t_0)\mathrm{d}t_2\right)^{a/b}}}, \end{cases} \tag{4.12}$$

$$\vdots$$

$$\vdots$$

$$\begin{cases} y_t(t_0) < \dfrac{1}{|b|\int_T^{t_0}\mathrm{d}tE_{12}(t;t_0)}, \\[2ex] y_b(t_0) < \dfrac{1}{|b|\int_T^{t_0}\mathrm{d}tE_{21}(t;t_0)} \end{cases}. \tag{4.13}$$

$$\tag{4.14}$$

The limit of this tower of inequalities (assuming of course that the iteration converges) is precisely the necessary and sufficient condition (4.9) and (4.10), as can be seen from (4.7) when written in a form similar to (2.14).

The sufficient conditions:

$$y_t(t_0) < \frac{1}{c(1+1/c)^{1+c}}\frac{E_1(t_0)}{|b|\int_T^{t_0}\mathrm{d}tE_1(t)}, \tag{4.15}$$

$$y_b(t_0) < \frac{1}{c(1+1/c)^{1+c}}\frac{E_2(t_0)}{|b|\int_T^{t_0}\mathrm{d}tE_2(t)}, \tag{4.16}$$

when $c = a/b$. It is interesting to note that the above conditions have basically the same form as (4.11), apart from the factor $1/(c(1+1/c)^{1+c})$ (~ 0.62 in the MSSM.)

5 Preliminary applications and comments

The aim of this section is to illustrate briefly through some examples the possible use of the integrated forms and to make contact with the existing studies and approximations. It is, however, obviously not meant to be exhaustive nor refined from the phenomenological point of view (for instance no threshold effects or higher loop effects are included), as this would deserve a separate analysis by itself. Let us also keep in mind that all the functions E_i which enter the solutions are analytically known in terms of the initial gauge coupling values as can been seen from (1.1) and its solutions.

5.1 Landau pole free bounds

The necessary and sufficient bounds found in the previous section are the exact generalization of the one in [2] initially derived in the regime of large y_t ($\gg y_b$). The necessary bounds (4.11) which would also be sufficient in the limit $a \to 0$ (where they become identical to (4.15) and (4.16)) give a first estimate of the allowed values for y_t, y_b at low energy, restricting them to a rectangle,

$$y_t^{\mathrm{MSSM}}(\mathrm{EW}) < \overline{y}_t \ , \ \ y_b^{\mathrm{MSSM}}(\mathrm{EW}) < \overline{y}_b, \tag{5.1}$$

where $\overline{y}_t$ and $\overline{y}_b$ depend on the gauge couplings and are easily determined numerically, for instance in terms of a grand unified scale, the value of the gauge couplings at that scale, and some low energy electroweak scale.

For instance, one has roughly (neglecting the g_1 and g_2 couplings)

$$\overline{y_{t,b}} = \frac{7}{18}\frac{g_3^4(\mathrm{EW})}{1-\left(\frac{g_3^2(\mathrm{EW})}{g_3^2(\mathrm{GUT})}\right)^{-7/9}}. \tag{5.2}$$

We stress that the necessary bounds in (5.1) involve no approximation whatsoever. One can readily turn them into constraints on the $\tan\beta$ parameter at some electroweak scale:

$$\frac{(\overline{m}_{\mathrm{top}}\overline{g}_2)^2}{(\sqrt{2\overline{y}_t}\overline{M}_W)^2-(\overline{m}_{\mathrm{top}}\overline{g}_2)^2} < \tan^2\beta < \frac{(\sqrt{2\overline{y}_b}\overline{M}_W)^2-(\overline{m}_b\overline{g}_2)^2}{(\overline{m}_b\overline{g}_2)^2}. \tag{5.3}$$

The bars indicate that the masses and gauge couplings are running quantities at the chosen low energy scale.

Going now to the improved bounds (4.12) one can reduce further the allowed range for the Yukawa couplings. These bounds do not allow in their general form an easy analytic determination of the allowed regions. To get a feeling about these regions, let us illustrate the case in two different approximations:

(1) $E_1 = E_2 \equiv E$, i.e. neglect the difference $f_1 - f_2 = g_1^2/24\pi^2$;
(2) assume $y_b(t_0), y_t(t_0)$ sufficiently small for a first order expansion to be legitimate

(1) In this case one integral can be performed exactly in (4.12), leading to

$$\begin{cases} y_t(t_0) < \dfrac{y_b(t_0)(1-\frac{a}{b})}{1-(1-|b|y_b(t_0)\int_T^{t_0}\mathrm{d}t_2E(t_2;t_0))^{1-a/b}}, \\ y_b(t_0) < \dfrac{y_t(t_0)(1-\frac{a}{b})}{1-(1-|b|y_t(t_0)\int_T^{t_0}\mathrm{d}t_2E(t_2;t_0))^{1-a/b}}. \end{cases} \tag{5.4}$$

The domain defined by (5.4) lies within the rectangle (5.1) and is controlled by the relative strength of a and b. For instance, the delimiting curves start off at the points $(\overline{y}_b, 0)$ and $(0, \overline{y}_t)$ with first derivatives equal to $-a/(2b)$ ($-1/12$ in the MSSM and $-1/6$ in the SM), to be compared with 0 in the rectangular approximation. Moreover, the exact fixed line solution $y_t = y_b$ leads to the constraint

$$y_t(t_0) = y_b(t_0) < \frac{E(t_0)}{|a+b|\int_T^{t_0}\mathrm{d}tE(t)}, \tag{5.5}$$

to be compared with the rectangular approximation (5.1) where $|a+b|$ is replaced by $|b|$, (a 14% effect in the MSSM and a 25% effect in the SM.) These considerations give a qualitative guideline of the reduction of the allowed domain.

(2) In this case one gets a linearized approximation of the domain [keeping though the effect of the $U(1)_Y$ coupling], in the following form,

$$\begin{cases} |b|(\int E_1)^2 y_t(t_0) + |a|(\int E_1 \int E_2)y_b(t_0) < \int E_1, \\ |b|(\int E_2)^2 y_b(t_0) + |a|(\int E_2 \int E_1)y_t(t_0) < \int E_1, \end{cases} \tag{5.6}$$

where the E_is here are normalized to $E_i(t_0)$, $\int\cdots \equiv \int_T^{t_0}\ldots \mathrm{d}t_1$ and $\int\cdots\int\cdots \equiv \int_T^{t_0}\ldots\mathrm{d}t_1\int_{t_1}^{t_0}\ldots\mathrm{d}t_2$.

In this approximation the necessary domain is delimited by two straight lines with scale dependent slopes. Again, one can translate these conditions into bounds on $\tan\beta$ at some effective electroweak scale.

5.2 $y_t - y_b - g_3$ approximation, fixed points and quasi-fixed line

In the approximation where $E_1(t) = E_2(t) \equiv E(t)$, and assuming that the initial values y_t^0, y_b^0 are small enough so that one iteration in the form of (2.12) and (2.13) is a good approximation for the $E_{ij}(t)$, an easy integration yields

$$y_t(t) = \frac{y_t^0 E(t)}{\left(1-by_b^0\int E\right)^{a/b}} \times \frac{1}{\left[1+\frac{y_t^0}{y_b^0}\frac{b}{b-a}\left(\left(1-by_b^0\int E\right)^{1-a/b}-1\right)\right]}, \tag{5.7}$$

$$y_b(t) = \frac{y_b^0 E(t)}{\left(1-by_t^0\int E\right)^{a/b}} \times \frac{1}{\left[1+\frac{y_b^0}{y_t^0}\frac{b}{b-a}\left(\left(1-by_t^0\int E\right)^{1-a/b}-1\right)\right]}, \tag{5.8}$$

provided of course all other Yukawas are put to zero. Note that at this level of approximations the above solutions depend just on one integral, namely $\int_0^t E$. If one goes further and neglects the $SU_{\mathrm{L}}(2)$ gauge coupling then $\int_0^t E$ can be computed explicitly, and one obtains

$$\rho_t(X) = \rho_t^0\frac{X}{[1+\alpha\rho_b^0(X-1)]^c} \times \frac{1}{\left[1+\frac{\rho_t^0}{\rho_b^0}\frac{1}{1-c}\left((1+\alpha\rho_b^0(X-1))^{1-c}-1\right)\right]}, \tag{5.9}$$

$$\rho_b(X) = \rho_b^0\frac{X}{[1+\alpha\rho_t^0(X-1)]^c} \times \frac{1}{\left[1+\frac{\rho_b^0}{\rho_t^0}\frac{1}{1-c}\left((1+\alpha\rho_t^0(X-1))^{1-c}-1\right)\right]} \tag{5.10}$$

where we now use the reduced variables

$$\rho_t \equiv \frac{y_t(t)}{g_3^2(t)} \ , \quad \rho_b \equiv \frac{y_b(t)}{g_3^2(t)}, \tag{5.11}$$

and

$$X \equiv E(t)^{7/16} = \Big(\frac{g_3^2(t)}{{g_3^0}^2}\Big)^{7/9}. \tag{5.12}$$

(Note that $\alpha = 18/7$, $c = a/b = 1/6$ in the MSSM and $\alpha = 9/2$, $c = 1/3$ in the SM.) The approximate solutions (5.9) and (5.10) allow one to retrieve the well-known infrared fixed points in the (ρ_t, ρ_b) plane [17]. For instance, in the MSSM, the fixed point $(\rho_t = 7/18, \rho_b = 0)$ (respectively $(\rho_t = 0, \rho_b = 7/18)$) is obtained by looking at the limiting behaviour of $\rho_t(X)$ when $\rho_b^0 \to 0$ (respectively of $\rho_b(X)$ when $\rho_t^0 \to 0$. On the other hand, the IR (attractive) fixed point $(\rho_t = 1/3, \rho_b = 1/3)$ is obtained by expanding $\rho_t(X)$ and $\rho_b(X)$ simultaneously for small ρ_t^0 and ρ_b^0. It should come as no surprise that those exact fixed points are obtained only in this limit since the solutions (5.9) and (5.10) become exact only in this limit. (See also a related comment at the end of this section.)

The exact fixed line $\rho_t = \rho_b$ is trivially obtained from (5.9) and (5.10), i.e. starting from $\rho_t^0 = \rho_b^0$ the two reduced Yukawas remain equal at any other scale. Perhaps it is more interesting to ask whether one can analytically determine the other exact or effective IR fixed lines. An IR attractive effective fixed line of the form

$$\rho_t + \rho_b = \frac{2}{3} \tag{5.13}$$

was found in [18] for the MSSM.

Starting from (5.9) and (5.10) one can actually improve on this effective fixed line in the following way. For small ρ_t^0, ρ_b^0 one obtains the integral line

$$\begin{aligned}\alpha(1-c)(\rho_b^0 - \rho_t^0)\rho_t\rho_b + \rho_t^0(1 - \alpha(\rho_b^0 + c\rho_t^0))\rho_b \\ - \rho_b^0(1 - \alpha(\rho_t^0 + c\rho_b^0))\rho_t = 0.\end{aligned} \tag{5.14}$$

If we require this integral line to go through the exact fixed point $\rho_t = \rho_b = 1/(\alpha(1+c))$ then a fixed line is obtained in the form

$$\frac{\rho_t + \rho_b}{\rho_t\rho_b} = 2(1+c)\alpha \tag{5.15}$$

to first order in an expansion around the fixed point $\rho_t = \rho_b = 1/(\alpha(1+c))$.

That is, we have

$$\frac{\rho_t + \rho_b}{\rho_t\rho_b} = 6 \tag{5.16}$$

in the case of the MSSM, which constitutes an improved effective IR fixed line beyond (5.13). One can even reasonably expect (5.16), and more generally (5.15), to be exact in the regime under consideration. Indeed, for instance in the case of the SM, the effective fixed line (5.15) reads

$$\frac{\rho_t + \rho_b}{\rho_t\rho_b} = 12. \tag{5.17}$$

On the other hand, an exact IR fixed line is known in this case (see [17]) and is the sum of two terms, one of which coincides precisely with (5.17), the other being vanishing to first order in the deviation, δ, around the fixed point, that is for $\rho_t = 1/6 + \delta, \rho_b = 1/6 - \delta$.

Let us end this section by noting further possible applications of the integrated forms. As we mentioned before, if one starts from (5.9) and (5.10), one retrieves the exact fixed point $\rho_t = \rho_b = 1/(\alpha(1+c))$ only in the region of small ρ_t^0, ρ_b^0. Moreover, in the deep infrared region ($X \to \infty$) one obtains from (5.9) and (5.10) $\rho_t = \rho_b = (1-c)/\alpha$ irrespective of the initial values ρ_t^0, ρ_b^0. The IR attraction to this point is of course an artifact of the approximate solutions. In fact, one can resum exactly the integrated forms (2.12) and (2.13) in the limit $X \to \infty$ and obtain $\rho_t = \rho_b = 1/(\alpha(1+c))$ as the attractive IR fixed point, independently of the initial values ρ_t^0, ρ_b^0.

Other regimes can be also looked at (for instance $\rho_t^0 \ll \rho_b^0$ or $\rho_b^0 \ll \rho_t^0$) for which approximate analytical expressions for the integrated forms can be obtained up to three iterations, thus improving on (5.9) and (5.10). This allows one to tackle the form of the fixed line in such regimes. We do not dwell further on these aspects here.

5.3 Constraints in the y_t–y_b–y_τ and all gauge couplings case

In this section we illustrate the use of the solutions in the t–b–τ system only to derive inequalities which correlate the three fermion running masses in terms of initial values for the Yukawa couplings. Starting from (2.15)–(2.16) and using the fact that $(1+|\alpha|y_i^0 \int u_i)^{|\beta|} \geq 1$ for all expressions of that form appearing in (2.15) and (2.15) one writes immediately the following (necessary) inequalities:

$$\overline{m}_t \leq Y_t^0 (2\sqrt{2}G_F)^{-1/2}\sqrt{E_1}\sin\beta, \tag{5.18}$$

$$\overline{m}_i \leq Y_i^0 (2\sqrt{2}G_F)^{-1/2}\sqrt{E_i}\cos\beta, \tag{5.19}$$

where $i = 2(b), 3(\tau)$ and recalling that the $y_i s$ are squares of the Yukawa couplings Y_i. Y_i^0 denotes the values of these couplings at some high energy scale and the bar denotes running quantities at the electroweak scale. The E_is are as defined in (2.9) and (2.16). These inequalities give boundary conditions on the initial values of the Yukawa couplings, necessary to retrieve the correct “physical” fermion masses. For instance one immediately sees from (5.19) that a large $\tan\beta$ necessitates large initial values for the bottom and τ Yukawa couplings.

Moreover, relying systematically on the fact that $(1 + |\alpha|y_i^0 \int u_i)^{|\beta|} \geq 1$ for $i = 1, 2, 3$, one can derive the following optimal rigorous inequalities for more involved combinations:

$$\frac{y_b^{18}}{y_t^3 y_\tau^{35}} \geq \frac{(y_b^0)^{18}}{(y_t^0)^3(y_\tau^0)^{35}} \frac{E_2^{18}}{E_1^3 E_3^{35}}, \quad (5.20)$$

$$\frac{y_t^2 y_\tau^3}{y_b^{12}} \geq \frac{(y_t^0)^2(y_\tau^0)^3}{(y_b^0)^{12}} \frac{E_1^2 E_3^3}{E_2^{12}}, \quad (5.21)$$

$$\frac{y_b^4}{y_t^{21} y_\tau} \geq \frac{(y_b^0)^4}{(y_t^0)^{21} y_\tau^0} \frac{E_2^4}{E_1^{21} E_3}, \quad (5.22)$$

which can be readily translated into inequalities involving the running quark masses, $\tan\beta$ and the three gauge couplings (all taken at the electroweak scale), as well as the values of the three Yukawa couplings at some initial scale

$$\frac{\overline{m_b}^{18}}{\overline{m_t}^3 \overline{m_\tau}^{35}} \geq \frac{(Y_b^0)^{18}}{(Y_t^0)^3(Y_\tau^0)^{35}} \frac{E_2^9}{\sqrt{E_1^3 E_3^{35}}} \frac{(2\sqrt{2}G_F)^{10}}{\sin^3\beta\cos^{17}\beta}, \quad (5.23)$$

$$\frac{\overline{m_t}^2 \overline{m_\tau}^3}{\overline{m_b}^{12}} \geq \frac{(Y_t^0)^2(Y_\tau^0)^3}{(Y_b^0)^{12}} \sqrt{E_3^3}\frac{E_1}{E_2^6} \frac{\tan^2\beta}{\cos^7\beta}(2\sqrt{2}G_F)^{7/2}, \quad (5.24)$$

$$\frac{\overline{m_b}^4}{\overline{m_t}^{21}\overline{m_\tau}} \geq \frac{(Y_b^0)^4}{(Y_t^0)^{21} Y_\tau^0} \frac{E_2^2}{\sqrt{E_1^{21} E_3}} \frac{(2\sqrt{2}G_F)^9}{\tan^3\beta\sin^{18}\beta}. \quad (5.25)$$

These inequalities express general necessary conditions which delineate the physically allowed regions for the initial values of the three Yukawa couplings, i.e. those which are consistent with the values of the physical top, bottom and τ masses, prior to any model assumption[4]. (Note also that (5.18) and (5.19) are already contained in (5.23)–(5.25).) Finally, if we had neglected the τ Yukawa coupling, the necessary inequalities involving the top and bottom running masses would have read

$$\frac{\overline{m_t}}{\overline{m_b}^6} \geq \frac{Y_t^0}{(Y_b^0)^6} \frac{\sqrt{E_1}}{E_2^9} \frac{\tan\beta}{\cos^5\beta}(2\sqrt{2}G_F)^{5/2}, \quad (5.26)$$

$$\frac{\overline{m_b}}{\overline{m_t}^6} \geq \frac{Y_b^0}{(Y_t^0)^6} \frac{\sqrt{E_2}}{E_1^9} \frac{1}{\tan\beta\sin^5\beta}(2\sqrt{2}G_F)^{5/2}. \quad (5.27)$$

The different origin and meaning of these inequalities as compared to (5.3) should be clear.

5.4 A numerical illustration

Even though the general form of the exact solutions is not directly exploitable analytically, truncated iterations provide very good approximations which furthermore can be very well controlled using the convergence criteria we derived. For instance, truncating at the first iteration, i.e. approximating E_{12} and E_{21} in ((2.12) and (2.13) by the explicit forms

$$E_{12}(t) \simeq \frac{E_1(t)}{(1 - b y_b^0 \int_0^t E_2(t')\mathrm{d}t')^{a/b}}, \quad (5.28)$$

$$E_{21}(t) \simeq \frac{E_2(t)}{(1 - b y_t^0 \int_0^t E_1(t')\mathrm{d}t')^{a/b}}, \quad (5.29)$$

and plugging them back into (2.10) and (2.11), one gets a simple analytical solution. In Table 1, a comparison is made for this solution with the Runge–Kutta method, showing an excellent agreement of less than 1% accuracy for any small, moderate or large values of $\tan\beta$.

Similar approximations can be obtained for the top–bottom–τ case, at least if the initial Yukawa couplings verify the sufficient convergence criterion of Sect. 3.2. We will not dwell on further possible applications in the present paper.

[4] In a more refined treatment, one should of course correct for the difference between the running and the pole masses

6 Conclusion

We have written down integrated forms for the running of the Yukawa couplings in the case of two and three Yukawa fermions, which are easily generalizable to any number of such fermions. These forms are *exact* solutions for the one-loop renormalization group equations, valid for virtually any gauge theory with a Yukawa sector. The most important feature of such forms is that they allow for a rigorous determination of convergence criteria as well as exact conditions for avoiding Landau-like poles of the Yukawa couplings. In the case of the MSSM, such criteria lead to approximate analytical solutions in the top–bottom system, with very good numerical accuracy ($\lesssim 1\%$) for any value of $\tan\beta$. Similar criteria were obtained for the top–bottom–τ system, which lead to controllable analytical approximations. In this context we gave some preliminary applications for Landau pole bounds in the top–bottom system, commented on some infrared fixed points and lines, and gave optimal necessary constraints on the Yukawa couplings both in the top–bottom and top–bottom–τ systems.

In view of the increasing phenomenological interest for the large $\tan\beta$ scenario, such solutions should prove useful in determining the exact structure of the running of the remaining parameters of the MSSM using for instance the method developed in [19], and possible implications on the structure of the (stable) infrared fixed points [20]. Very recently, the authors of reference [21] have addressed similar issues, starting though from approximate solutions.

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A Necessary and sufficient conditions for non-singular evolutions

A.1 A necessary condition

A necessary condition not to meet a singularity in running *up* from an initial energy scale $t = t_0$ to a given high energy scale $t = T < t_0$ is easy to establish. Indeed, if $y_{b,t}(t)$ are free from singularities in the interval $[T, t_0]$, then $E_{ij}(t; t_0)$ are necessarily positive for any t in this interval, as can be

Table 1. Numerical comparison between the exact one-loop solution (truncated to the first iteration) and the Runge–Kutta RG evolution. The evolution is over 10 orders of magnitude starting from the initial Yukawa coupling values shown in the table

$\tan\beta$	Y_b^0	Y_t^0	$Y_b(t)$, truncated	$Y_b(t)$, R.–K.	$Y_t(t)$, truncated	$Y_t(t)$, R.–K.
2	0.0387453	1.13007	0.0145059	0.0145050	0.775788	0.775974
10	0.174138	1.01581	0.0630978	0.0631052	0.54263	0.542743
50	0.866544	1.01097	0.435682	0.439526	0.585453	0.590258

seen from (4.6) and the fact that $b < 0$ and $y_{b,t}(t) > 0$. It then follows from (4.7) that

$$E_{12}(t, t_0) \geq E_1(t, t_0) \quad \text{and} \quad E_{21}(t, t_0) \geq E_2(t, t_0), \quad \text{(A.1)}$$

since the denominators in (4.7) are always smaller than 1 (recall that $a/b > 0$ and $t < t_0$). From the above considerations one immediately gets the inequalities

$$\begin{aligned} 1 - |b| y_t(t_0) & \int_t^{t_0} \mathrm{d}t' E_{12}(t'; t_0) \\ & \leq 1 - |b| y_t(t_0) \int_t^{t_0} \mathrm{d}t' E_1(t'; t_0), \qquad \text{(A.2)} \\ 1 - |b| y_b(t_0) & \int_t^{t_0} \mathrm{d}t' E_{21}(t'; t_0) \\ & \leq 1 - |b| y_b(t_0) \int_t^{t_0} \mathrm{d}t' E_2(t'; t_0), \qquad \text{(A.3)} \end{aligned}$$

for any t in the interval $[T, t_0]$. Again, from (4.6), the left-hand side of (A.2) should remain positive for any t in the interval $[T, t_0]$, for $y_t(t)$ is free from singularities there. In particular, $t = T$ gives the most significant condition, whence

$$1 - |b| y_t(t_0) \int_T^{t_0} \mathrm{d}t' E_1(t'; t_0) > 0, \qquad \text{(A.4)}$$

which is the necessary condition given in (4.11). One similarly obtains the second inequality in (4.11).

On the other hand, from (A.3) and the positivity of its left-hand side one gets

$$\begin{aligned} E_{12}(t; t_0) &= \frac{E_1(t; t_0)}{\left(1 - b y_b(t_0) \int_{t_0}^t E_{21}(t'; t_0) \mathrm{d}t'\right)^{a/b}} \\ &\geq \frac{E_1(t; t_0)}{\left(1 - b y_b(t_0) \int_{t_0}^t E_2(t'; t_0) \mathrm{d}t'\right)^{a/b}}, \quad \text{(A.5)} \end{aligned}$$

valid since E_1 and a/b are both positive. One can thus repeat the proof which led to (A.4) with $E_1(t; t_0)$ replaced by

$$\frac{E_1(t; t_0)}{\left(1 - b y_b(t_0) \int_{t_0}^t E_2(t'; t_0) \mathrm{d}t'\right)^{a/b}}$$

in (A.1) to get the first inequality in (4.12), and the second in a similar way. The infinite tower of inequalities (4.11)–(4.14) is obtained recursively in the same way.

A.2 A sufficient condition

Similarly to what was done in Sect. 3, (4.7) defines a mapping $\mathcal{A}$ in the form

$$\begin{aligned} e'_{12}(t) &= \frac{1}{\left(1 - |b| y_b^0 \int_t^{t_0} e_2(t') e_{21}(t') \mathrm{d}t'\right)^{a/b}}, \\ e'_{21}(t) &= \frac{1}{\left(1 - |b| y_t^0 \int_t^{t_0} e_1(t') e_{12}(t') \mathrm{d}t'\right)^{a/b}}, \qquad \text{(A.6)} \end{aligned}$$

where

$$\begin{aligned} e_{ij}(t) &\equiv \frac{E_{ij}(t; t_0)}{E_i(t; t_0)}, & \text{(A.7)} \\ e_i(t) &\equiv E_i(t; t_0), & \text{(A.8)} \\ y_{t,b}^0 &\equiv y_{t,b}(t_0). & \text{(A.9)} \end{aligned}$$

Again, we collect the $e_{ij}(t)$'s in a vector

$$\vec{E}(t) = \begin{pmatrix} e_{12}(t) \\ e_{21}(t) \end{pmatrix}, \qquad \text{(A.10)}$$

and consider the range $T \leq t \leq t_0$ where t_0 corresponds to some low energy scale at which initial values for y_t, y_b are chosen, and T to a high energy scale (typically a GUT scale) up to which we require the Yukawa couplings to have a non-singular behaviour. We also define a norm similar to (3.5):

$$\| \vec{E} \| = \max \left\{ \sup_{T \leq t \leq t_0} | e_{12}(t) |, \sup_{T \leq t \leq t_0} | e_{21}(t) | \right\}. \quad \text{(A.11)}$$

To determine the conditions we are looking for to avoid singularities in the range $[T, t_0]$ it will actually suffice to ask: when does the mapping defined in (A.6) become a contraction? Parts of the proof will resemble that of Sect. 3.1. However, in contrast to the latter case, where the mapping defined in (3.3) could not have singularities as long as $U_1(t), U_2(t) \geq 0$, in the present case one has to make sure that the mapping $\mathcal{A}$ keeps $e_{ij}(t)$ within a finite interval, $1 \leq e_{ij}(t) \leq R$.

For a given R, let us thus denote by X_R the set of all vectors $\vec{E}(t)$ such that $1 \leq e_{ij}(t) \leq R$ for any t in the interval $[T, t_0]$. We look for conditions on the values of R, y_t^0, y_b^0 such that

(1) The mapping $\mathcal{A}$ sends any element of X_R in X_R (so that the $e_{ij}(t)$'s remain in the interval $[1, R]$ after an arbitrary number of iterations of $\mathcal{A}$);

(2) $\mathcal{A}$ is a strictly contracting mapping in X_R, that is

$$\| \mathcal{A}(\vec{E}_1) - \mathcal{A}(\vec{E}_2) \| \leq K_R \| \vec{E}_1 - \vec{E}_2 \|, \qquad (A.12)$$

with some $K_R < 1$.

Condition (i) means that

$$e'_{ij}(t) = \frac{1}{\left(1 - |b| y^0_{b,t} \int_t^{t_0} e_j(t') e_{ji}(t') \mathrm{d}t'\right)^{a/b}} \leq R. \quad (A.13)$$

On the other hand, one finds from $e_{ij}(t') \leq R$ that

$$e'_{ij} \leq \frac{1}{\left(1 - |b| y^0_{b,t} R \int_t^{t_0} e_j(t') \mathrm{d}t'\right)^{a/b}}, \qquad (A.14)$$

provided that $1 - |b| y^0_{b,t} R \int_t^{t_0} e_j(t') \mathrm{d}t'$ is a positive number[5].

In view of (A.13) and (A.14) a sufficient condition to obtain (i) is

$$\frac{1}{\left(1 - |b| y^0_t R \int_t^{t_0} e_1(t') \mathrm{d}t'\right)^{a/b}} \leq R,$$

and

$$\frac{1}{\left(1 - |b| y^0_b R \int_t^{t_0} e_2(t') \mathrm{d}t'\right)^{a/b}} \leq R,$$

which easily translates into

$$y_1 \leq \frac{1}{R} - \frac{1}{R^{1+1/c}},$$
$$y_2 \leq \frac{1}{R} - \frac{1}{R^{1+1/c}}, \qquad (A.15)$$

where $c \equiv a/b$ and

$$y_{\frac{1}{2}} \equiv |b| y^0_{\frac{t}{b}} \int_T^{t_0} e_{\frac{1}{2}}(t) \mathrm{d}t. \qquad (A.16)$$

At this level R is still an arbitrary number. However, the optimal situation would be to choose it such that the upper bound (A.15) becomes the largest possible. This would be the case for $R = (1 + 1/c)^c$, but one still has to check whether this value is compatible with the second requirement (ii) which we turn to now.

Using the inequality (3.7), one gets from (A.6)

$$\begin{aligned}
&| e'_{12}(t) - \tilde{e}'_{12}(t) | \\
&\leq \left\{ c|b| y^0_b \int_T^{t_0} \mathrm{d}t' e_2(t') \mid e_{21}(t') - \tilde{e}_{21}(t') \mid \right\} \\
&\Big/ \left\{ \left[1 - |b| y^0_b \max \left\{ \int_T^{t_0} \mathrm{d}t' e_2(t') e_{21}(t'), \right.\right.\right. \\
&\left.\left.\left. \int_T^{t_0} \mathrm{d}t' e_2(t') \tilde{e}_{21}(t') \right\} \right]^{1+c} \right\} \\
&\leq \frac{c|b| y^0_b \int_T^{t_0} \mathrm{d}t' e_2(t')}{\left[1 - |b| y^0_b R \int_T^{t_0} e_2(t') \mathrm{d}t' \right]^{1+c}} \| \vec{E} - \tilde{\vec{E}} \|, \quad (A.17)
\end{aligned}$$

valid for any t in the interval $[T, t_0]$ and $\vec{E}, \tilde{\vec{E}}$ belonging to X_R. A similar inequality holds obviously for $| e'_{21}(t) - \tilde{e}'_{21}(t) |$, and one finally gets

$$\| \mathcal{A}(\vec{E}) - \mathcal{A}(\tilde{\vec{E}}) \| \leq K_R \| \vec{E} - \tilde{\vec{E}} \|, \qquad (A.18)$$

with

$$K_R = \max \left\{ \frac{c y_2}{(1 - R y_2)^{1+c}}, \frac{c y_1}{(1 - R y_1)^{1+c}} \right\}, \qquad (A.19)$$

where y_1, y_2 are defined in (A.16). It is now easy to check that when condition (A.15) is satisfied with strict inequalities, one gets

$$K_R < 1, \qquad (A.20)$$

even for the value of R quoted before, $R_0 = (1 + 1/c)^c$, which maximizes the bounds in (A.15). Since (A.18) and (A.20) mean that the mapping is indeed contracting, one concludes that the sufficient conditions for (i) given in (A.15) with maximal bounds, i.e.

$$y_1 < \frac{1}{c(1 + 1/c)^{1+c}},$$
$$y_2 < \frac{1}{c(1 + 1/c)^{1+c}}, \qquad (A.21)$$

imply also (ii). It follows that when (A.21) (equivalently (4.15) and (4.16)) are satisfied, a unique, regular, solution for (4.7) exists in X_{R_0}. The regularity of y_t, y_b as given by (4.6) is then an immediate consequence.

B: Approximate solutions for $y_t \gg y_b \neq 0$

To first order in y^0_b one finds for y_t

$$y_t(t) = \frac{y^0_t E_1(t)}{1 - b y^0_t F_1(t)} \left[1 + \frac{a y^0_b}{1 - b y^0_t F_1(t)} \times \int_0^t \frac{E_2(t')}{(1 - b y^0_t F_1(t'))^{a/b - 1}} \mathrm{d}t' \right], \qquad (B.1)$$

where

$$F_1(t) \equiv \int_0^t E_1(t') \mathrm{d}t', \qquad (B.2)$$

and where the solution for y_b is given in (2.7).

C: Exact integrated forms for an arbitrary number of Yukawa couplings

Under the restriction of flavour conserving Yukawa couplings, and assuming that the Higgs fields sit in representations such that the renormalization group equations for

[5] This condition will, however, turn out to be already contained in the sufficient condition we are looking for

the Yukawa couplings can be cast in the following form at the one-loop level [16]:

$$\frac{\mathrm{d}}{\mathrm{d}t} y_i = y_i \left(f_i(t) + \sum_j a_{ij} y_j \right), \qquad \text{(C.1)}$$

where i, j count the fermion fields, and y_i denotes the square of the ith Yukawa coupling.

Then the exact solution for each y_i reads

$$y_i(t) = \frac{y_i^0 u_i}{1 - a_{ii} y_i^0 \int_0^t u_i}, \qquad \text{(C.2)}$$

where the u_is are given by the implicit equations

$$u_i(t) = \frac{E_i(t)}{\prod\limits_{j \neq i} (1 - a_{jj} y_j^0 \int_0^t u_j)^{a_{ij}/a_{jj}}}, \qquad \text{(C.3)}$$

and $E_i(t) = \mathrm{e}^{\int_0^t f_i(t')\mathrm{d}t'}$.

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